

Ornstein's non-inequalities Riesz product approach

Krystian Kazaniecki

Michał Wojciechowski

July 1, 2014

Abstract

We provide a new technique to prove Ornstein's non-inequalities for derivatives with some geometrical dependence on their indexes.

In [2] D. Ornstein studied behavior of norms of the homogeneous differential operators with the same degree of homogeneity. He proved lack of a priori estimates for linearly independent differential operators. We show alternative way of proving Ornstein's non-inequalities in some special cases.

Theorem 1. *Assume $\alpha_0, \dots, \alpha_n$ are multindexes in \mathbb{R}^d . If there exists a pair of vectors $\Gamma, \Lambda \in (\mathbb{N} \cup \{0\})^d$ for which following occurs*

$$\langle \alpha_0, \Lambda \rangle = \langle \alpha_1, \Lambda \rangle = \dots = \langle \alpha_m, \Lambda \rangle,$$

and

$$\langle \alpha_0, \Gamma \rangle < \langle \alpha_1, \Gamma \rangle < \langle \alpha_2, \Gamma \rangle \leq \dots \leq \langle \alpha_m, \Gamma \rangle,$$

Then for every $K > 0$ there exists $f \in C^\infty(\mathbb{T}^d)$ such that

$$\|D^{\alpha_1} f\|_{L_1(\mathbb{T}^d)} \geq K \sum_{j \neq 1} \|D^{\alpha_j} f\|_{L_1(\mathbb{T}^d)}.$$

Instead of giving full proof, we limit ourself to the special, yet representative, case. We prove that for every $K > 0$ there exists $f \in C^\infty(\mathbb{T}^d)$ such that

$$\left\| \frac{\partial^5}{\partial x_1^3 \partial x_2^2} f \right\|_{L_1(\mathbb{T}^2)} \geq K \left(\left\| \frac{\partial^4}{\partial x_1^4} f \right\|_{L_1(\mathbb{T}^2)} + \left\| \frac{\partial^6}{\partial x_1^2 \partial x_2^4} f \right\|_{L_1(\mathbb{T}^2)} + \left\| \frac{\partial^7}{\partial x_1 \partial x_2^6} f \right\|_{L_1(\mathbb{T}^2)} + \left\| \frac{\partial^8}{\partial x_2^8} f \right\|_{L_1(\mathbb{T}^2)} \right). \quad (1)$$

Proof. We fix $K > 0$ and $n > 64K^2C^{-2}$. We will construct trigonometric polynomial, whose one of the derivatives behaves like the modified Riesz product

$$R_n(x) = -1 + \prod_{k=1}^n (1 + \cos(2\pi \langle x, a_k \rangle)),$$

where $a_k \in \mathbb{Z}^2$. By induction we can choose $a_k \in \mathbb{Z}^2$ for $k = 1, \dots, n$ such that

$$\left| \left(\frac{(a_k(2) + \sum_{j=1}^l \epsilon_j a_j(2))^2}{a_k(1) + \sum_{j=1}^l \epsilon_j a_j(1)} \right)^l - \left(\frac{\sigma_k}{\sqrt{n}} \right)^l \right| \leq \frac{1}{3^n}, \quad (2)$$

$$|a_{k+1}| > M_n |a_k|,$$

$$a_k(1) + \sum_{j=1}^l \epsilon_j a_j(1) \neq 0,$$

for every $k \in \{1, 2, \dots, n\}$, $l \in \{1, 2, \dots, m\}$ and $\epsilon_j \in \{-1, 0, 1\}$, where $n > (8K + 1)^2 C^{-2}$ and the values of parameters M_n , C and σ_j are determined by the following lemma.

Lemma 1. (cf. [3]) *There is $C > 0$ such that for every $m \in \mathbb{N}^+$ there exists $M = M(m)$ and a sequence $\{\sigma_j\}_{j=1}^m \in \{0, 1\}^m$ such that*

$$\left\| \sum_{j=1}^m \sigma_j \cos(2\pi \langle d_j, \xi \rangle) \prod_{1 \leq k < j} (1 + \cos(2\pi \langle d_k, \xi \rangle)) \right\|_{L_1(\mathbb{T}^d)} \geq Cn$$

whenever $\{d_k\}_{k=1}^m \subset \mathbb{Z}^d$ satisfies $|d_{k+1}| > M_m |d_k|$ for $k = 1, \dots, m-1$.

This inequality was generalized by R. Latała in [1]. We define family of sets

$$A_k = \{q : q = a_k + \sum_{j=1}^{k-1} \epsilon_j a_j, \epsilon_j = -1, 0, 1\}.$$

For $q = \sum_{j=1}^n \epsilon_j(q) a_j$ we put $r(q) = \#\{j : \epsilon_j \neq 0\}$. Let Z be the polynomial given by the formula

$$Z(x) = \sum_{k=1}^n \sum_{q \in A_k \cup -A_k} \frac{1}{q(1)^4} \frac{1}{2^{r(q)}} e^{2i\pi \langle q, x \rangle}.$$

Simple calculation gives

$$D^{\alpha_0} Z(x) = \sum_{k=1}^n \sum_{q \in A_k \cup -A_k} \frac{1}{2^{r(q)}} e^{2i\pi \langle q, x \rangle} = R_n(x).$$

Hence

$$\|D^{\alpha_0} Z_n(x)\|_{L_1(\mathbb{T}^2)} \leq 2$$

Since $D^{\alpha_l} Z(x) = D^{\alpha_l - \alpha_0} D^{\alpha_0} Z(x) = D^{l(\alpha_1 - \alpha_0)} R_n$, for $m \geq 1$ we have

$$i^{-l} D^{\alpha_l} Z(x) = \sum_{k=1}^n \sum_{q \in A_k \cup -A_k} \left(\frac{q(2)^2}{q(1)} \right)^l \frac{1}{2^{r(q)}} e^{2i\pi \langle q, x \rangle} = I_l + II_l,$$

where

$$I_l = \sum_{k=1}^n \sum_{q \in A_k \cup -A_k} \left(\frac{q(2)^{2l}}{q(1)^l} - \left(\frac{\sigma_k}{\sqrt{n}} \right)^l \right) \frac{1}{2^{r(q)}} e^{2i\pi \langle q, x \rangle},$$

$$II_l = \sum_{k=1}^n \sum_{q \in A_k \cup -A_k} \left(\frac{\sigma_k}{\sqrt{n}} \right)^l \frac{1}{2^{r(q)}} e^{2i\pi \langle q, x \rangle}.$$

Since (2) is satisfied and $\#A_k = 3^{k-1}$,

$$\|I_l\|_{L_1(\mathbb{T}^2)} \leq 1$$

It is easy to check that

$$II_l = n^{-\frac{l}{2}} \sum_{k=1}^n \sigma_k^l \cos(2\pi \langle a_k, \xi \rangle) \prod_{1 \leq j < k} (1 + \cos(2\pi \langle a_j, \xi \rangle)).$$

By the triangle inequality for $m \geq 2$ we have

$$\|II_m\|_{L_1(\mathbb{T}^2)} \leq n^{-\frac{m}{2}} n \leq 1$$

For $m = 1$ by Lemma 1, we get

$$\|II_1\|_{L_1(\mathbb{T}^2)} \geq Cn n^{-\frac{1}{2}} = C\sqrt{n} > 8K + 1,$$

which proves (1). □

References

- [1] Rafał Łatała. L_1 -norm of combinations of products of independent random variables. *Israel J. Math.*, to appear.
- [2] Donald Ornstein. A non-equality for differential operators in the L_1 norm. *Arch. Rational Mech. Anal.*, 11:40–49, 1962.
- [3] Michał Wojciechowski. On the strong type multiplier norms of rational functions in several variables. *Illinois J. Math.*, 42(4):582–600, 1998.